A lower bound on the order of the largest induced forest in planar graphs with high girth *

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Abstract

We give here new upper bounds on the size of a smallest feedback vertex set in planar graphs with high girth. In particular, we prove that a planar graph with girth g and size m has a feedback vertex set of size at most $\frac{4m}{3g}$, improving the trivial bound of $\frac{2m}{g}$. We also prove that every 2-connected graph with maximum degree 3 and order n has a feedback vertex set of size at most $\frac{n+2}{3}$.

1 Introduction

In this article we only consider finite simple graphs.

Let G be a graph. A *feedback vertex set* or *decycling set* S of G is a subset of the vertices of G such that removing the vertices of S from G yields an acyclic graph. Thus S is a feedback vertex set of G if and only if the graph induced by $V(G)\backslash S$ in G is an induced forest of G. The FEEDBACK VERTEX SET DECISION PROBLEM (which consists of, given a graph G and an integer k, deciding whether there is a decycling set of G of size k) is known to be NP-complete, even restricted to the case of planar graphs, bipartite graphs or perfect graphs [10]. It is thus legitimate to seek bounds for the size of a decycling set or an induced forest. The smallest size of a decycling set of G is called the *decycling number* of G, and the highest order of an induced forest of G is called the *forest number* of G, denoted respectively by $\phi(G)$ and $a(G)$. Note that the sum of the decycling number and the forest number of G is equal to the order of G (i.e. $|V(G)| = a(G) + \phi(G)$).

Mainly, the community focuses on the following challenging conjecture due to Albertson and Berman [3]:

Conjecture 1 (*Albertson and Berman [3]*)**.** *Every planar graph G of order n admits an induced forest of order at least $\frac{n}{2}$, that is $a(G) \geq \frac{n}{2}$.*

Conjecture 1, if true, would be tight (for $n \geq 3$ multiple of 4) because of the disjoint union of complete graphs on four vertices (Akiyama and Watanabe [1] gave examples showing that the conjecture differs from the optimal by at most one half for all n), and would imply that every planar graph has an independent set on at least a quarter of its vertices, the only

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Figure 1: An outerplanar graph G with $a(G) = \frac{2|V(G)|}{3}$.

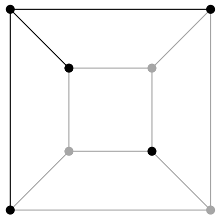

Figure 2: The cube Q admits an induced forest on five of its vertices, but no induced forest on six or more of its vertices, i.e. $a(Q) = 5$.

known proof of which relies on the Four-Color Theorem. The best known lower bound to date for the forest number of a planar graph is due to Borodin and is a consequence of the acyclic 5-colorability of planar graphs [6]. We recall that an acyclic k-coloring is a proper vertex coloring using k colors such that the graph induced by the vertices of any two color classes is a forest. From Borodin's result one can obtain the following theorem:

Theorem 2 (*Borodin [6]*)**.** *Every planar graph of order n admits an induced forest of order at least $\frac{2n}{5}$.*

Hosono [9] showed the following theorem and showed that the bound is tight.

Theorem 3 (*Hosono [9]*)**.** *Every outerplanar graph of order n admits an induced forest of order at least $\frac{2n}{3}$.*

The tightness of the bound is shown by the example in Figure 1.

Akiyama and Watanabe [1], and Albertson and Haas [2] independently raised the following conjecture:

Conjecture 4 (*Akiyama and Watanabe [1], and Albertson and Haas [2]*)**.** *Every bipartite planar graph of order n admits an induced forest of order at least $\frac{5n}{8}$.*

This conjecture, if true, would be tight for n multiple of 8: for example, if G is the disjoint union of k cubes, then we have $a(G) = 5k$ and G has order $8k$ (see Figure 2). Motivated by Conjecture 4, Alon [4] proved the following theorem using probabilistic methods:

Theorem 5 (*Alon [4]*)**.** *There exist some absolute constants $b > 0$ and $b' > 0$ such that:*

- *For every bipartite graph G with n vertices and average degree at most d (≥ 1), $a(G) \geq (\frac{1}{2} + e^{-bd^2})n$.*
- *For every $d \geq 1$ and all sufficiently large n, there exists a bipartite graph with n vertices and average degree at most d such that $a(G) \leq (\frac{1}{2} + e^{-b'\sqrt{d}})n$.*

The lower bound was later improved by Conlon et al. [7] to $a(G) \geq (1/2 + e^{-b''d})n$ for a constant b''.

Conjecture 4 also led to researches for lower bounds of the forest number of triangle-free planar graphs (as a superclass of bipartite planar graphs). Alon *et al.* [5] proved the following theorem and corollary:

Theorem 6 (*Alon et al. [5]*)**.** *Every triangle-free graph of order n and size m admits an induced forest of order at least $n - \frac{m}{4}$.*

Corollary 7 (*Alon et al. [5]*)**.** *Every triangle-free cubic graph of order n admits an induced forest of order at least $\frac{5n}{8}$.*

Theorem 6 is tight because of the union of cycles of length 4.

The *girth* of a graph is the length of a shortest cycle. A forest has infinite girth. In a planar graph with girth at least g, order n, and size m with at least one cycle, the number of faces is at most $2m/g$ (since all the faces' boundaries have length at least g). Then, by Euler's formula, $2m/g \geq m - n + 2$, and thus $m \leq (g/(g-2))(n-2)$. In particular, triangle-free planar graphs of order $n \geq 3$ have size at most $2n - 4$. As a consequence of Theorem 6, for a triangle-free planar graph G of order n, $a(G) \geq n/2$. Salavatipour proved a better lower bound [12]: $a(G) \geq \frac{17n+24}{32}$. In a companion paper, the authors strengthen this bound as follows:

Theorem 8 (*[8]*)**.** *Every triangle-free planar graph of order $n \geq 1$ admits an induced forest of order at least $\frac{6n+7}{11}$.*

Kowalik *et al.* [11] made the following conjecture on planar graphs of girth at least 5:

Conjecture 9 (*Kowalik et al. [11]*)**.** *Every planar graph with girth at least 5 and order n admits an induced forest of order at least $7n/10$.*

This conjecture, if true, would be tight for n multiple of 20, as shown by the example of the union of dodecahedrons, given by Kowalik *et al.* [11] (see Figure 3).

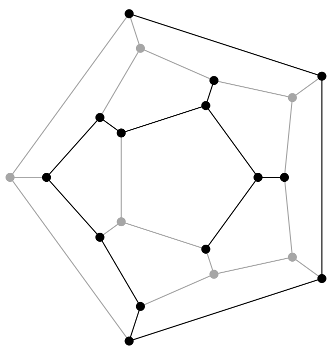

Figure 3: The dodecahedron D admits an induced forest on fourteen of its vertices, but no induced forest on fifteen or more of its vertices, i.e. $a(D) = 14$.

A first step toward Conjecture 9 was done in a companion paper [8]; moreover a generalization for higher girth was given:

Theorem 10 (*[8]*)**.** *Every planar graph with girth at least 5 and order $n \geq 1$ admits an induced forest of order at least $\frac{44n+50}{69}$.*

Theorem 11 (*[8]*)**.** *Every planar graph with girth at least $g \geq 5$ and order $n \geq 1$ admits an induced forest of order at least $n - \frac{(5n-10)g}{23(g-2)}$.*

For planar graphs with given girth, we conjecture the following:

Conjecture 12. *Let G be a planar graph of size m and girth g. There exists a feedback vertex set S of G of size at most $\frac{m}{g}$.*

Planar graph with girth g	Forest number
4	$\frac{6n+7}{11}$
5	$\frac{44n+50}{69}$
6	$\frac{31n+30}{46}$
$g \geq 7$	$\frac{(3g-10)n+8}{3(g-2)}$

Table 1: Lower bounds on the forest numbers for planar graphs with given girth.

If Conjecture 12 is true, then it is tight for m multiple of g due to the union of disjoint cycles of length g. It is easy to prove that G admits a feedback vertex set of size at most $\frac{2m}{g}$ (removing a vertex that is in the boundary of at least two faces decreases the number of faces by one, and this can be applied recursively).

The main result of this paper is a first non-trivial step toward Conjecture 12:

Theorem 13. *Let G be a planar graph of size m and girth g. There exists a feedback vertex set S of G of size at most $\frac{4m}{3g}$.*

Theorem 13 is the best result so far for $g \geq 7$, and gives $a(G) \geq \frac{(3g-10)n+8}{3(g-2)}$ using $m \leq (n-2)\frac{g}{g-2}$ (Theorem 11 is better for $g = 6$). We summarize the previous results in Table 1.

Theorem 13 will be proven in Section 3. For this, we will use Theorem 14 (proven in Section 2) that is of independent interest. Let $\mathcal{C}_{2,3^-}$ be the familly of 2-connected graphs of maximum degree at most 3.

Theorem 14. *Every graph in $\mathcal{C}_{2,3^-}$ of order n has a feedback vertex set of size at most $\frac{n+2}{3}$.*

Theorem 14 is tight for the complete graph on 4 vertices. Moreover, consider any 3-regular graph G, and consider the graph H obtained from G by replacing each vertex by a triangle (as the cube connected cycles obtained from the hypercube). Graph H has $3|V(G)|$ vertices, and cannot have a feedback vertex set of less than $|V(G)|$ vertices, since a feedback vertex set of H contains at least one vertex of each added triangle. Hence there is a graph of order n without a feedback vertex set of size less than $\frac{n}{3}$ for an arbitrary large n.

Finally, if we replace the 2-connected condition by simply connected, then $\frac{3n}{8} + \frac{1}{4}$ becomes a tight bound [5]. One can observe that without connected condition, the disjoint union of complete graphs on four vertices has a smallest feedback vertex set of size $\frac{n}{2}$.

Notations. Consider $G = (V, E)$. For a set $S \subseteq V$, let $G - S$ be the graph obtained from G by removing the vertices of S and all the edges that are incident to a vertex of S. If $x \in V$, then we denote $G - \{x\}$ by $G - x$. For a set S of vertices such that $S \cap V = \emptyset$, let $G + S$ be the graph constructed from G by adding the vertices of S. If $x \notin V$, then we denote $G + \{x\}$ by $G + x$. For a set F of pairs of vertices of G such that $F \cap E = \emptyset$, let $G + F$ be the graph constructed from G by adding the edges of F. If e is a pair of vertices of G and $e \notin E$, then we denote $G + \{e\}$ by $G + e$. For a set $W \subseteq V$, we denote by $G[W]$ the subgraph of G induced by W. We call a vertex of degree d, at least d, and at most d, a *d-vertex*, a *d^+-vertex*, and a *d^--vertex* respectively. Similarly, we call a cycle of length ℓ, at least ℓ, and at most ℓ a *ℓ-cycle*, a *ℓ^+-cycle*, and a *ℓ^--cycle* respectively, and by extension a face of length ℓ, at least ℓ, and at most ℓ a *ℓ-face*, a *ℓ^+-face*, and a *ℓ^--face* respectively. For a face f of a plane graph G, we denote the boundary of f by $G[f]$. We say that two faces are *adjacent* if their boundaries share (at least) an edge. We say that two cycles are *adjacent* if they share at least an edge. An *edge cut-set* of a graph G is a minimal set of edges F such that $G \backslash F$ is disconnected. If an edge cut-set is a singleton, then its element is a *cut edge*. A *vertex cut-set* of a graph G is a set X of vertices of G such that $G \backslash X$ is disconnected. If a vertex cut-set is a singleton, then its element is a *cut vertex*.

2 Proof of Theorem 14

We recall that $G = (V, E)$ is called *k-connected* if $|V| > k$ and $G - X$ is connected for every set $X \subseteq V$ with $|X| < k$. Also $G = (V, E)$ is called *k-edge connected* if $|V| > 1$ and the deletion of any set of at most $(k-1)$ edges leads to a connected graph.

Let us consider $H = (V, E)$ a counter-example to Theorem 14 of minimum order, and $n = |V| \geq 3$ be the order of H. Let us prove some lemmas on the structure of H.

Lemma 15. *Graph H is cubic.*

Proof. Suppose there is a vertex v of degree at most 2 in H. As H is 2-connected, v has degree 2. Let u and w be the two neighbors of v in H. Suppose $uw \notin E$. Let $H' = H - v + uw$. Since u and w have degree at least 2 (H is 2-connected), $|V(H')| \geq 3$. Then graph H' is in $\mathcal{C}_{2,3^-}$ since H is. By minimality of H, H' has a feedback vertex set S of size $|S| \leq \frac{n-1+2}{3} \leq \frac{n+2}{3}$, and S is also a feedback vertex set of H, a contradiction. Therefore $uw \in E$. If both u and w have degree 2, then $H = C_3$ and H admits a feedback vertex set of size $1 \leq \frac{n+2}{3} = \frac{5}{3}$, a contradiction. If one of u and w has degree 2 and the other one has degree 3, then H is not 2-connected, a contradiction. Therefore both u and w have degree 3. Let u' and w' be the third neighbors of u and w respectively. If $u' = w'$, then $V = \{u, v, w, u'\}$ (H is 2-connected), and H admits a feedback vertex set of size $1 \leq \frac{n+2}{3} = 2$ ($\{u\}$ for example), a contradiction. Thus u' and w' are distinct. Suppose $u'w' \in E$. Let $H' = H - \{u, v, w\}$. If $|V(H')| < 3$, then u' and w' are adjacent vertices of degree 2 in H and we fall into a previous case. Therefore $|V(H')| \geq 3$. Then graph H' is in $\mathcal{C}_{2,3^-}$ since H is. By minimality of H, H' has a feedback vertex set S' of size $|S'| \leq \frac{n-3+2}{3}$. The set $S = S' \cup \{u\}$ is a feedback vertex set of H of size $|S| \leq \frac{n-3+2}{3} + 1 = \frac{n+2}{3}$, a contradiction. Therefore $u'w' \notin E$. Let $H' = H - \{u, v, w\} + u'w'$. Graph H' is in $\mathcal{C}_{2,3^-}$ since H is. By minimality of H, H' has a feedback vertex set S' of size $|S'| \leq \frac{n-3+2}{3}$. The set $S = S' \cup \{u\}$ is a feedback vertex set of H of size $|S| \leq \frac{n-3+2}{3} + 1 = \frac{n+2}{3}$, a contradiction. □

In the following, we will use the fact that H is cubic without referring to Lemma 15.

Lemma 16. *There are no adjacent triangles in H.*

Proof. Assume that there are two triangles xyz and xyz' sharing an edge xy in H. If $zz' \in E$, then $H = K_4$ (H is connected), which contradicts the fact that H is a counter-example to Theorem 14. Therefore $zz' \notin E$. Let v be the neighbor of z distinct from x and y. Observe that $vz' \notin E$, since H is cubic and 2-connected. Let $H' = H - \{x, y, z\} + vz'$. Graph H' is in $\mathcal{C}_{2,3^-}$ since H is. By minimality of H, H' has a feedback vertex set S' of size $|S'| \leq \frac{n-3+2}{3}$. The set $S = S' \cup \{x\}$ is a feedback vertex set of H of size $|S| \leq \frac{n-3+2}{3} + 1 = \frac{n+2}{3}$, a contradiction. □

Lemma 17. *There is no triangle that shares an edge with a 4-cycle in H.*

Proof. By Lemma 16, there is no triangle that shares two edges with a 4-cycle in H. Assume that there are a triangle xyw and a 4-cycle $vzxy$ that share the edge xy.

Suppose first that there is a vertex z' adjacent to v and w. If $zz' \in E$, then $V = \{v, w, x, y, z, z'\}$ (H is connected), i.e. H is the prism, and $\{y, z\}$ is a feedback vertex set of H, thus H is not a counter-example to Theorem 14, a contradiction. Therefore $zz' \notin E$. Let z'' be the third neighbor of z'. Let $H' = H - \{w, y, z'\} + xv + vz''$. Graph H' is in $\mathcal{C}_{2,3^-}$ since H is. By minimality of H, H' admits a feedback vertex set S' of size at most $|S'| \leq \frac{n-3+2}{3}$. The set $S = S' \cup \{w\}$ is a feedback vertex set of H of size $|S| \leq \frac{n-3+2}{3} + 1 = \frac{n+2}{3}$, a contradiction.

Therefore there is no vertex adjacent to v and w. Let w' be the neighbor of w distinct from x and y. Let $H'' = H - \{x, y, w\} + vw'$. Graph H'' is in $\mathcal{C}_{2,3^-}$. By minimality of H, H'' admits a feedback vertex set S'' of size at most $|S''| \leq \frac{n-3+2}{3}$. The set $S = S'' \cup \{x\}$ is a feedback vertex set of H of size $|S| \leq \frac{n-3+2}{3} + 1 = \frac{n+2}{3}$, a contradiction. □

Lemma 18. *There are no two* 4*-cycles that share two edges in* H*.*

Proof. Let $uvwx$ and $vwxy$ be two 4-cycles of H. Let u', w' and y' be the third neighbors of u, w and y respectively. By Lemma 16, they are distinct from the vertices defined previously. If $u' = w' = y'$, then $H = K_{3,3}$ admits a feedback vertex set of size $2 \leq \frac{6+2}{3} = \frac{n+2}{3}$ (for example $\{u, y\}$), a contradiction.

Suppose $u' \neq w' \neq y' \neq u'$. Let $H' = H - \{u, v, w, y\} + \{u'x, w'x, y'x\}$. If H' is not 2-connected, then w.l.o.g. x separates u' and w' in H', and thus u separates u' and w' in H, a contradiction. Therefore H' is in $\mathcal{C}_{2,3^-}$. By minimality of H, H' admits a feedback vertex set S' of size at most $\frac{n-4+2}{3}$. The set $S = S' \cup \{v\}$ is a feedback vertex set of H of size $|S'| + 1 \leq \frac{n-4+2}{3} + 1 \leq \frac{n+2}{3}$, a contradiction.

Thus w.l.o.g., $u' = y' \neq w'$. Let z be the neighbor of u' distinct from u and y. Observe that z is distinct from w' since H is cubic and 2-connected. Let $H' = H - \{u, v, w, x, y, u'\}$ if $zw' \in E$ and $H' = H - \{u, v, w, x, y, u'\} + zw'$ otherwise. Graph H' is in $\mathcal{C}_{2,3^-}$ since H is. By minimality of H, H' admits a feedback vertex set S' of size at most $\frac{n-6+2}{3}$. The set $S = S' \cup \{v, x\}$ is a feedback vertex set of H of size $|S'| + 2 \leq \frac{n-6+2}{3} + 2 \leq \frac{n+2}{3}$, a contradiction. ◻

Lemma 19. *For every $k \in \{1, 2, 3\}$, a graph with maximum degree at most* 3 *is k-connected if and only if it is k-edge-connected.*

Proof. Let G be a graph with maximum degree at most 3. One can easily check that the result holds for the complete graph on at most four vertices.

Suppose now that G is not complete. Let C_v be a vertex cut-set of G and C_e be a edge cut-set of G, both of minimum size. If we show that $|C_v| = |C_e|$, then the lemma holds.

Let V_1 and V_2 be the vertex sets of the two connected components of $G - C_e$. We have $V_1 \cup V_2 = V(G)$. By minimality of $|C_e|$, every edge of C_e has an endvertex in V_1 and the other one in V_2. Suppose every vertex of V_1 is adjacent to every vertex of V_2 in G. We have $|C_e| = |V_1||V_2| \geq |V_1| + |V_2| - 1 = |V(G)| - 1$. Moreover, for any vertex in G, the set of the edges incident to this vertex is an edge cut-set of G. Therefore, since G is not complete, by minimality of C_e, $|C_e| \leq |V(G)| - 2$, a contradiction. Therefore there are two vertices $v_1 \in V_1$ and $v_2 \in V_2$ such that $v_1v_2 \notin E(G)$. Let $C'_v = \{x \neq v_1 | \exists y \in V_2, xy \in C_e\} \cup \{y | v_1y \in C_e\}$. Note that $|C'_v| = |\{x \neq v_1 | \exists y \in V_2, xy \in C_e\}| + |\{y | v_1y \in C_e\}| \leq |C_e|$. For each edge in C_e, one of the endvertices of this edge is in C'_v. As neither v_1 nor v_2 is in C'_v, C'_v separates v_1 from v_2 in G. Therefore $|C_v| \leq |C'_v|$, and thus $|C_v| \leq |C_e|$.

Let W_1 and W_2 be the vertex sets of two connected components of $G - C_v$. Let $x \in C_v$. Since x has degree at most 3, x has at most one neighbor in W_1 or at most one neighbor in W_2, and it has at least one neighbor in W_1 and one in W_2 by minimality of C_v. Let y_x be the neighbor of x that is in W_1 if there is only one neighbor of x in W_1, and the neighbor of x in W_2 otherwise, and $e_x = xy_x$. Observe that this defines a unique edge e_x for every $x \in C_v$. Let $C'_e = \{e_x | x \in C_v\}$. Assume C'_e does not separate W_1 and W_2. There are $v_1 \in W_1$ and $v_2 \in W_2$ such that there is a path P from v_1 to v_2 in $H - C'_e$. Let us consider v_1 and v_2 such that P has minimal length. Then there are w_1 and w_2 in C_v such that $v_1w_1 \in E(P)$ and $v_2w_2 \in E(P)$. If $w_1 = w_2$, then either $v_1w_1 \in C'_e$ or $v_2w_2 \in C'_e$, a contradiction. If $w_1 \neq w_2$, then w_1 has a neighbor in $V(G) \backslash (W_1 \cup W_2)$, so it has only one neighbor in W_1, that is v_1, so $v_1w_1 \in C'_e$, a contradiction. Therefore C'_e separates W_1 and W_2. We have $|C'_e| = |C_v|$, thus $|C_e| \leq |C_v|$. Finally, since $|C_v| \leq |C_e|$, $|C_v| = |C_e|$. ◻

Lemma 20. *Graph H is* 3*-connected.*

Proof. Suppose by contradiction that H is not 3-connected. By Lemma 15, $|V(H)| \geq 4$. By hypothesis and Lemma 19, H is 2-edge-connected but not 3-edge-connected. Let $\{e, f\}$ be an edge cut-set of H that induces two connected components V_1 and V_2 such that $|V_1|$ is minimum.

We will now prove the two following properties:

- P_e: The deletion of any edge in $H[V_1]$ preserves the 2-edge connectivity of H.

  By contradiction, suppose there is an edge e' that has both of its endvertices in V_1 such that $H - e'$ is not 2-edge-connected (but connected since H is 2-edge-connected). Let f' be a cut edge of $H - e'$. If f' has at least one of its endvertices in V_1, then one of the connected components of $H - \{e', f'\}$ is strictly included in V_1, a contradiction with the minimality of $|V_1|$. Therefore, f' has both of its endvertices in V_2. Neither e nor f is a cut edge of $H - e'$, otherwise we fall into the previous case. Thus e' is not a cut edge of $H[V_1]$. In particular, there is a path in $H \backslash \{f', e'\}$ that connects the two endvertices of e'. However, e' is a cut edge of $H \backslash f'$, a contradiction.

- P_v: For every vertex v in V_1 that has all of its neighbors in V_1, $H - v$ is 2-edge-connected, and thus 2-connected by Lemma 19.

  Suppose there is a vertex $v \in V_1$ that is not incident to an edge of $\{e, f\}$ such that $H - v$ is not 2-edge-connected. Let f' be a cut edge of $H - v$. As vertex v has degree 3, there is an edge e' incident to v such that $H - \{e', f'\}$ is disconnected. As v is not incident to an edge of $\{e, f\}$, e' has both of its endvertices in V_1, a contradiction with P_e.

Let $v \in V_1$ and $u \in V_2$ such that $e = uv$. Let w and x be the two neighbors of v distinct from u. Vertices w and x are in V_1, otherwise w.l.o.g. $f = vw$, and vx is a cut edge of H, a contradiction.

Let us show that $wx \notin E$. By contradiction assume that $wx \in E$. Let w' be the neighbor of w distinct from v and x, and x' be the neighbor of x distinct from v and w. By Lemmas 16 and 17, w', x' and u are distinct and pairwise not adjacent. Moreover, if $w' \notin V_1$ or $x' \notin V_1$, say $w' \notin V_1$, then $f = ww'$, and thus xx' is a cut edge of H, a contradiction. Hence v, w, x, w' and x' are all in V_1, and thus, by P_v, $H - w$ is 2-connected. Let $H' = H - \{v, w, x\} + ux'$. Graph H' is in $\mathcal{C}_{2,3^-}$. By minimality of H, H' admits a feedback vertex set S' of size at most $\frac{n-3+2}{3}$. The set $S = S' \cup \{w\}$ is a feedback vertex set of H of size $|S'| + 1 \leq \frac{n-3+2}{3} + 1 \leq \frac{n+2}{3}$, a contradiction.

Let w_0 and w_1 be the two neighbors of w distinct from v. If w_0 or w_1 is in V_2, say $w_0 \in V_2$, then $ww_0 = f$, and $\{vx, ww_1\}$ is an edge cut-set of H, contradicting the minimality of $|V_1|$. Therefore w_0 and w_1 are in V_1.

Let us show that $w_0 w_1 \notin E$. By contradiction assume that $w_0 w_1 \in E$. Let w'_0 be the neighbor of w_0 distinct from w and w_1, and w'_1 be the neighbor of w_1 distinct from w and w_0. By Lemmas 16 and 17, w'_0 and w'_1 are distinct and not adjacent. Vertices v, w, w_0 and w_1 are all in V_1, thus, by P_v, $H - w$ is 2-connected. Let $H' = H - \{w, w_0, w_1\} + w'_0 w'_1$. Graph H' is in $\mathcal{C}_{2,3^-}$. By minimality of H, H' admits a feedback vertex set S' of size at most $\frac{n-3+2}{3}$. The set $S = S' \cup \{w\}$ is a feedback vertex set of H of size $|S'| + 1 \leq \frac{n-3+2}{3} + 1 \leq \frac{n+2}{3}$, a contradiction.

Let w_{00} and w_{01} be the two neighbors of w_0 distinct from w. Let us show that $w_{00} w_{01} \notin E$. By contradiction assume that $w_{00} w_{01} \in E$. Let w'_{00} be the neighbor of w_{00} distinct from w_0 and w_{01}, and w'_{01} be the neighbor of w_{01} distinct from w_0 and w_{00}. By Lemmas 16 and 17, w'_{00} and w'_{01} are distinct and not adjacent. Suppose w_{00} or w_{01} is in V_2, say $w_{00} \in V_2$. Then $w_0 w_{00} = f$, and e, f is not an edge cut-set of H (since $w_0 w_{00} w_{01}$ is a triangle), a contradiction. Therefore w, w_0, w_{00} and w_{01} are in V_1, and thus, by P_v, $H - w_0$ is 2-connected. Let $H' = H - \{w_0, w_{00}, w_{01}\} + w'_{00} w'_{01}$. Graph H' is in $\mathcal{C}_{2,3^-}$. By minimality of H, H' admits a feedback vertex set S' of size at most $\frac{n-3+2}{3}$. The set $S = S' \cup \{w_0\}$ is a feedback vertex set of H of size $|S'| + 1 \leq \frac{n-3+2}{3} + 1 \leq \frac{n+2}{3}$, a contradiction.

Let w_{10} and w_{11} be the two neighbors of w_1 distinct from w. By symmetry, $w_{10} w_{11} \notin E$. Suppose $\{w_{00}, w_{01}\} = \{w_{10}, w_{11}\}$; say $w_{00} = w_{10}$ and $w_{01} = w_{11}$. Lemma 18 leads to a contradiction. Therefore the pairs $\{w_{00}, w_{01}\}$ and $\{w_{10}, w_{11}\}$ are not equal. As v, w, w_0 and w_1 are in V_1, by P_v, $H - w$ is 2-connected. Let $H' = H - \{w, w_0, w_1\} + \{w_{00} w_{01}, w_{10} w_{11}\}$. Graph H' is in $\mathcal{C}_{2,3^-}$. By minimality of H, H' admits a feedback vertex set S' of size at most

$\frac{n-3+2}{3}$. The set $S = S' \cup \{w\}$ is a feedback vertex set of H of size $|S'|+1 \leq \frac{n-3+2}{3}+1 \leq \frac{n+2}{3}$, a contradiction, which completes the proof. $\square$

Lemma 21. *There is no triangle in H.*

Proof. Suppose there is a triangle uvw in H. Let u', v' and w' be the third neighbor of u, v and w respectively. By Lemmas 16 and 17, u', v' and w' are distinct and non-adjacent. Let $H' = H - \{u, v, w\} + u'v'$. Observe that by Lemma 20, $H - w$ is 2-connected. Therefore H' is in $\mathcal{C}_{2,3^-}$. By minimality of H, H' admits a feedback vertex set S' of size at most $\frac{n-3+2}{3}$. The set $S = S' \cup \{w\}$ is a feedback vertex set of H of size $|S'| + 1 \leq \frac{n-3+2}{3} + 1 \leq \frac{n+2}{3}$, a contradiction. $\square$

Let v be a vertex of H, and x and y be two neighbors of v. They are not adjacent by Lemma 21. Let x_0, x_1, y_0 and y_1 the two other neighbors of x and y respectively. Vertices x_0 and x_1 are not adjacent by Lemma 21, and similarily y_0 and y_1 are not adjacent. The pairs $\{x_0, x_1\}$ and $\{y_0, y_1\}$ are distinct by Lemma 18. Let $H' = H - \{v, x, y\} + \{x_0x_1, y_0y_1\}$. By Lemma 20, H' is in $\mathcal{C}_{2,3^-}$. By minimality of H, H' admits a feedback vertex set S' of size at most $\frac{n-3+2}{3}$. The set $S = S' \cup \{v\}$ is a feedback vertex set of H of size $|S'| + 1 \leq \frac{n-3+2}{3} + 1 \leq \frac{n+2}{3}$, a contradiction. That completes the proof of Theorem 14.

3 Proof of Theorem 13

Let $g \geq 3$ be a fixed integer. For G a planar graph, $\omega : E(G) \to \mathbb{N}$ a weight function, and $F \subseteq E(G)$, we denote $\sum_{e \in F}(\omega(e))$ by $\omega(F)$, and $\sum_{e \in E(G)}(\omega(e))$ by $\omega(G)$. We will prove the following claim:

Claim 22. *Let G be a planar graph, and $\omega : E(G) \to \mathbb{N}$ a weight function such that for each cycle C of G, $\omega(C) \geq g$. There exists a feedback vertex set S of G of size at most $\frac{4\omega(G)}{3g}$.*

Observe that fixing ω constant equal to 1 in Claim 22 yields Theorem 13. Let us consider any embedding of the graph G in the plane.

Let G be a 2-connected plane graph. Three faces f_0, f_1 and f_2 of G are said to be *mergeable* if:

1. there exists a vertex v that is in the boundary of f_0, f_1 and f_2.
2. w.l.o.g. f_0 and f_1 (resp f_1 and f_2) have at least one common edge in their boundary.

Given three mergeable faces f_0, f_1 and f_2, the *merger* of f_0, f_1 and f_2 consists in removing the edges belonging to the boundary of two faces among f_0, f_1 and f_2 as well as the vertices that end up being isolated. The common vertex v of f_0, f_1 and f_2 is called the *crucial* vertex of the merger. A merger is *nice* if the sum of the weights of the edges removed is at least $\frac{3g}{4}$. Observe that a merger cannot decrease $\min_{C \text{ cycle of } G}(\omega(C))$, since we only delete vertices and edges. See Figure 4 for an example of the merger of three faces.

Lemma 23. *Let G be a 2-connected plane graph, and G' obtained from G by applying a merger of crucial vertex v. If S' is a feedback vertex set of G', then $S' \cup \{v\}$ is a feedback vertex set of G.*

Proof. Let C be a cycle of G that contains an edge $e \in E(G) \backslash E(G')$. Edge e is in the boundary of two of the faces that are merged, say f_0 and f_1. Cycle C separates f_0 and f_1. Therefore it contains all the vertices of $V(G[f_0]) \cap V(G[f_1])$. In particular, it contains v.

Therefore each cycle of G is either entirely in G', or it contains v. Thus as $V(G') \backslash S'$ induces a forest in G', $V(G) \backslash (S' \cup \{v\})$ induces a forest in G. $\square$

Lemma 24. *Let G be a 2-connected plane graph, and G' be obtained from G by applying a nice merger. If graph G' satisfies Claim 22, then graph G also satisfies Claim 22.*

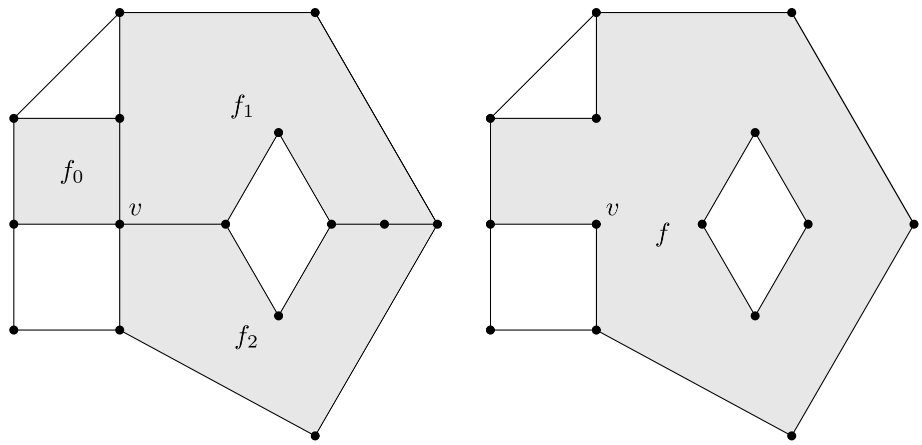

Figure 4: The merger of faces f_0, f_1 and f_2 into f with crucial vertex v.

Proof. Let v be the crucial vertex of the merger. We have $\omega(G') \leq \omega(G) - \frac{3g}{4}$. Since G' verifies Claim 22, there exists a feedback vertex set S' of G' such that $|S'| \leq \frac{4\omega(G')}{3g} \leq \frac{4\omega(G)}{3g} - 1$. Then $S = S' \cup \{v\}$ is a feedback vertex set of G (by Lemma 23), and $|S| \leq \frac{4\omega(G)}{3g} - 1 + 1 = \frac{4\omega(G)}{3g}$, which completes the proof. □

Let us assume by contradiction that there are couples (G, ω) that do not satisfy Claim 22. Among all counterexamples (G, ω) to Claim 22 minimizing $\omega(G)$, we consider a couple (G, ω) minimizing $\sum_{v \in V(G)}(\max\{0.5, d(v) - 2.5\})$.

Lemma 25. *Graph G is 2-connected.*

Proof. By contradiction, assume G is not 2-connected. Graph G has at least 2 vertices, otherwise it would satisfy Claim 22. Let S be a minimal vertex cut-set of G. We have $|S| \leq 1$. Let V_1 and V_2 be non-empty sets of vertices separated by S.

Let $\omega_1 = \omega(G[V_1 \cup S])$ and $\omega_2 = \omega(G[V_2 \cup S])$. By minimality of (G, ω), let $S_1 \subseteq V_1 \cup S$ and $S_2 \subseteq V_2 \cup S$ be feedback vertex sets of $V_1 \cup S$ and $V_2 \cup S$ respectively, such that $|S_1| \leq \frac{4\omega_1}{3g}$ and $|S_2| \leq \frac{4\omega_2}{3g}$. Now $S_1 \cup S_2$ is a feedback vertex set of G, and $|S_1 \cup S_2| \leq \frac{4\omega_1}{3g} + \frac{4\omega_2}{3g} = \frac{4\omega(G)}{3g}$. Thus G satisfies Claim 22, a contradiction. □

Lemma 26. *No nice mergers can be done in G.*

Proof. It follows from Lemma 24 and the minimality of (G, ω). □

Lemma 27. *Every face in G has at least three 3^+-vertices in its boundary.*

Proof. Let us assume that there is a face f in G with at most two 3^+-vertices in its boundary. Face f is adjacent to at most two other faces in G. Suppose f is adjacent to exactly one face, say f'. As G is 2-connected by Lemma 25, $G[f]$ and $G[f']$ are cycles. As f is adjacent only to f', $E(G[f]) \subseteq E(G[f'])$, and thus $G[f] = G[f']$. So two faces of G have exactly the same boundary, so G is a cycle, and it satisfies Claim 22, a contradiction.

Thus f is adjacent to exactly two other faces, say f_0 and f_1. Then $E(G[f]) \subseteq E(G[f_0]) \cup E(G[f_1])$, and $E(G[f]) \cap E(G[f_0]) \neq \emptyset \neq E(G[f]) \cap E(G[f_1])$. As $G[f]$ is a cycle, there is a vertex v in $V(G[f])$ incident to an edge in $E(G[f]) \cap E(G[f_0])$ and to an edge in $E(G[f]) \cap E(G[f_1])$. Merging the faces f, f_0 and f_1 with crucial vertex v is nice, since

we remove all the edges of $G[f]$ and $\omega(f) \geq g \geq \frac{3g}{4}$. This leads to a contradiction with Lemma 26. $\square$

Lemma 28. *There are no 4^+-vertices in G.*

Proof. Suppose v is a d-vertex in G with $d \geq 4$. Let u_0, ..., u_{d-1} be the neighbors of v. Let $G' = G - v + \{w, w'\} + \{wu_0, wu_1, ww', w'u_2, ..., w'u_{d-1}\}$, $\omega(wu_0) = \omega(vu_0)$, $\omega(wu_1) = \omega(vu_1)$, $\omega(w'u_2) = \omega(vu_2)$, ... , $\omega(w'u_{d-1}) = \omega(vu_{d-1})$, and $\omega(ww') = 0$. See Figure 5 for an illustration of this construction. Clearly, $\omega(G') = \omega(G)$. As we removed a d-vertex, added a 3-vertex and a $(d-1)$-vertex, and did not change the degree of the other vertices, $\sum_{v\in V(G')}(\max\{0.5, d(v) - 2.5\}) = \sum_{v\in V(G)}(\max\{0.5, d(v) - 2.5\}) - 0.5$.

It is easy to see that for any cycle C' of G', there is a cycle in G that has the same weight, so $\omega(C') \geq g$.

By minimality of (G, ω), let S' be a feedback vertex set of G' with $|S'| \leq \frac{4\omega(G')}{3}$. For any cycle C of G there is a cycle C' of G' such that $C = C'$ or $V(C) = (V(C')\backslash\{w, w'\}) \cup \{v\}$. If $w \in S'$ or $w' \in S'$, then let $S = S'\backslash\{w, w'\} \cup \{v\}$ and otherwise let $S = S'$. Then $|S| \leq |S'| \leq \frac{4\omega(G')}{3} = \frac{4\omega(G)}{3}$, and S is a feedback vertex set of G, a contradiction. $\square$

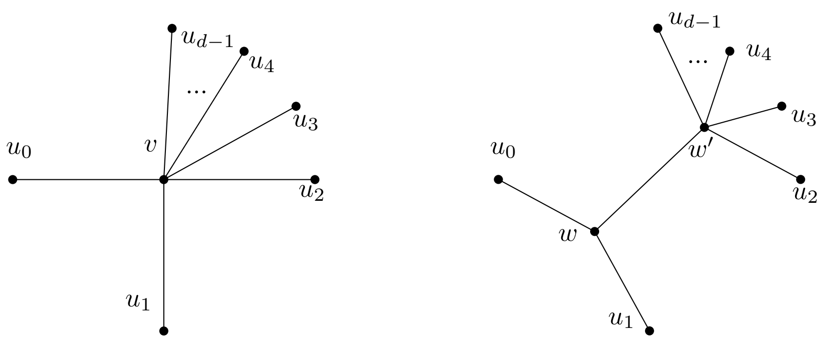

Figure 5: The construction of Lemma 28.

Lemma 29. *Every cycle has at least three 3-vertices in G.*

Proof. Let C be a cycle of G. By Lemma 28, every vertex in $V(C)$ has degree at most 3. Suppose C is a separating cycle. By Lemma 25, graph G is 2-connected, so at least two vertices of $V(C)$ have a neighbor in the interior of C, and at least two vertices of $V(C)$ have a neighbor in the exterior of C. Therefore C has at least four 3-vertices. Now if C bounds a face, then Lemma 27 concludes the proof. $\square$

Lemma 30. *Graph G is cubic (i.e. 3-regular).*

Proof. Suppose v is a 2^--vertex in G. Vertex v has degree 2 by Lemma 25. Let u and w be the two neighbors of v. By lemma 29, $uw \notin E(G)$.

Let $G' = G - v + uw$ and $\omega(uw) = \omega(uv) + \omega(vw)$. See Figure 6 for an illustration of this construction. Clearly, $\omega(G') = \omega(G)$. As we removed a 2-vertex and did not change the degree of the other vertices, $\sum_{v\in V(G')}(\max\{0.5, d(v) - 2.5\}) = \sum_{v\in V(G)}(\max\{0.5, d(v) - 2.5\}) - 0.5$.

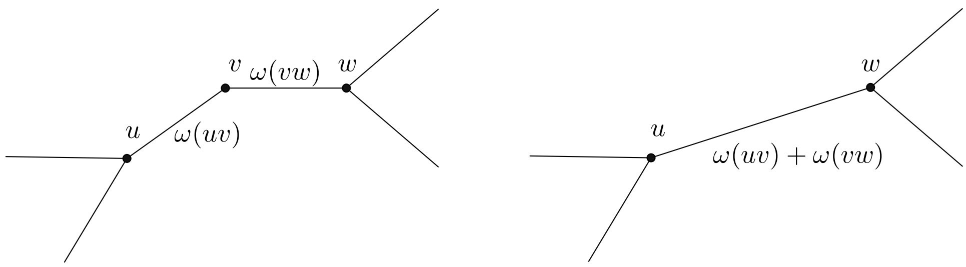

Figure 6: The construction of Lemma 30.

Let C' be any cycle of G'. If $uw \notin E(C')$, then C' is a cycle of G, and so $\omega(C') \geq g$. Otherwise, $C = C' - uw + v + \{uv, vw\}$ is a cycle of G, and $\omega(C) = \omega(C')$, so $\omega(C') \geq g$.

For any cycle C of G there is a cycle C' of G' that contains all the vertices of $V(C)\backslash\{v\}$. By minimality of (G, ω), let S' be a feedback vertex set of G' with $|S'| \leq \frac{4\omega(G')}{3} = \frac{4\omega(G)}{3}$. The set S' is a feedback vertex set of G, a contradiction. $\square$

By Lemmas 25 and 30, graph G is a 2-connected cubic graph. By Theorem 14, G admits a feedback vertex set of order at most $\frac{|V(G)|+2}{3}$. Let us denote by n the order of G, by m the size of G and by f the number of faces of G.

By Euler's formula, we have $n - m + f = 2$. We have $3n = 2m$ as G is cubic. Therefore, $f = 2 + m - n = 2 + \frac{n}{2}$, i.e. $n = 2(f - 2)$. Therefore G has a feedback vertex set S of size $|S| \leq \frac{2f-4+2}{3} \leq \frac{2f}{3}$. As each face has weight at least g, we have $gf \leq 2\omega(G)$, so $|S| \leq \dfrac{4\omega(G)}{3g}$, a contradiction, completing the proof of Theorem 13.

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